

4.1 Classical Probability

In classical probability, we assume that all outcomes are _____.

example: **flipping a coin...** $P(\text{heads}) = \underline{\hspace{2cm}}$ $P(\text{tails}) = \underline{\hspace{2cm}}$

rolling a die... $P(4) = \underline{\hspace{2cm}}$ $P(\text{odd}) = \underline{\hspace{2cm}}$ $P(7) = \underline{\hspace{2cm}}$

Formula for Classical Probability

The probability of any event E is

$$\frac{\text{Number of outcomes in } E}{\text{Total number of outcomes in the sample space}}$$

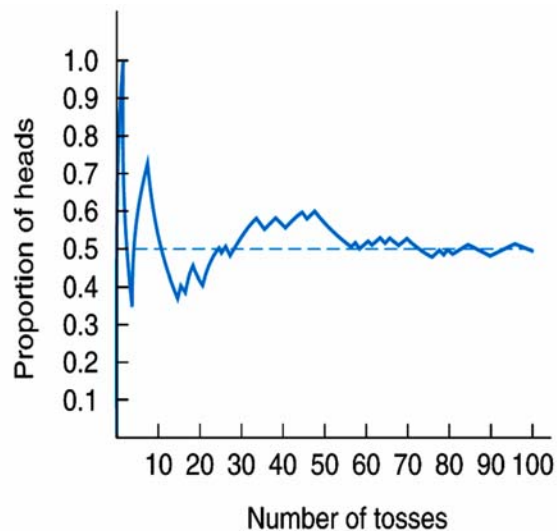
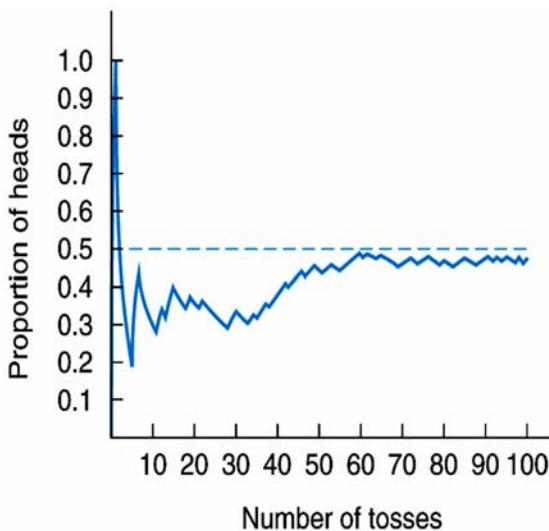
This probability is denoted by

$$P(E) = \frac{n(E)}{n(S)}$$

This probability is called *classical probability*, and it uses the sample space S .

- BASIC PROPERTIES:**
1. $P(E)$ is always between $\underline{\hspace{1cm}}$ and $\underline{\hspace{1cm}}$.
 2. The probability of an impossible event is $\underline{\hspace{1cm}}$.
 3. The probability of a certain event is $\underline{\hspace{1cm}}$.

The **frequential interpretation of probability** construes the proportion of times it occurs in a large number of repetitions of the event.



Two computer simulations of tossing a balanced coin 100 times

Dice Chart:

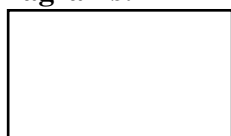
	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

P(2)=_____ P(7)=_____ P(multiple of 5)=_____

Sample Space -

For any event E, there is a corresponding event defined by the condition "E does not occur." It is called the **complement of E** and is denoted by "**not E**."

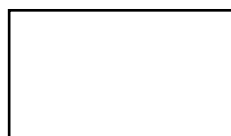
Venn Diagrams:



not E



A & B



A or B

Definitions: Suppose A and B are events.

not A: the event that "A does not occur"

A&B: the event that **both** event A and event B occur

A or B: the event that **either** event A **or** event B occur

Example: A={1,2,3} B={1,3,5} C={4,5,6}

$A \cup B$ _____ $A \cup C$ _____ $A \cap B$ _____ $A \cap C$ _____

example: A die is tossed. Consider the following events:

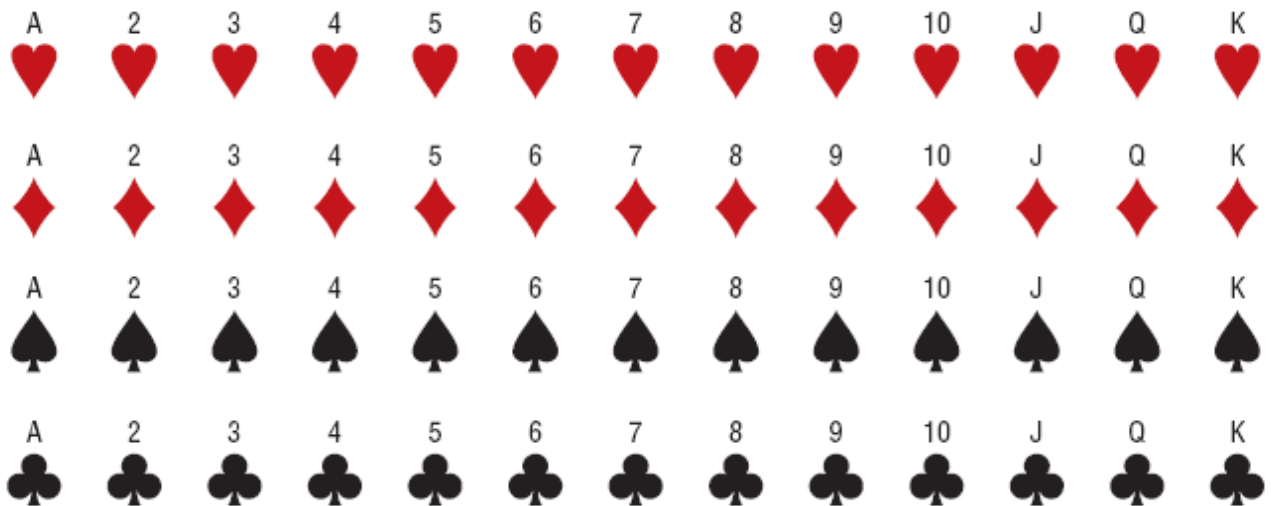
A= the event that an even is rolled

B= the event that an odd is rolled

C= the event that a 1, 2, or 3 is rolled.

List the outcomes which comprise each event:

A&B _____ **A&C** _____ **not C** _____ **A or B** _____ **A or C** _____



example: Consider a shuffled deck of 52 cards and the following events:

- A**= the event that a club is chosen
- B**= the event that a face card is chosen
- C**= the event that the 6 of spades is chosen
- D**= the event that a 6 is chosen

Find the following probabilities:

P(A)= _____ **P(B)**= _____ **P(C)**= _____ **P(D)**= _____

Describe the following in words:

not A:

A & D:

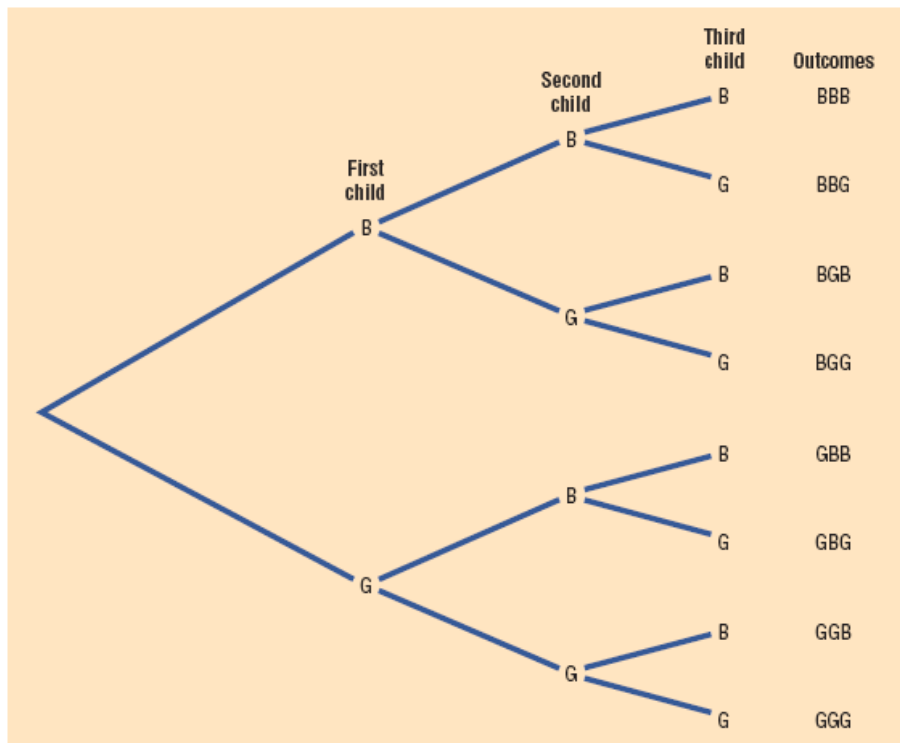
A or C:

A **tree diagram** is a device consisting of line segments emanating from a starting point and also from the outcome point. It is used to determine all possible outcomes of a probability experiment.

Use a tree diagram to find the sample space for the gender of three children in a family, as in Example 4–3.

Solution

Since there are two possibilities (boy or girl) for the first child, draw two branches from a starting point and label one B and the other G. Then if the first child is a boy, there are two possibilities for the second child (boy or girl), so draw two branches from B and label one B and the other G. Do the same if the first child is a girl. Follow the same procedure for the third child. The completed tree diagram is shown in Figure 4–3. To find the outcomes for the sample space, trace through all of the possible branches, beginning at the starting point for each one.



The **odds** that an event occurs can be found using the ratio of the number of ways it *can* occur to the number of ways it *cannot* occur:

Example: Find the odds of rolling a two with a single die.

Example: A class contains 18 men and 14 women.

- a) Find the probability of choosing a woman at random.

- b) Find the odds of choosing a woman at random.

4.2 The Addition Rules for Probability

Two events are said to be _____ if they cannot both occur at the same time.

A collection of _____ and _____ events occur if:

- a) each event is mutually exclusive of all others; and
- b) the union of the events is the sample space.

Example:

Addition Rule: $P(A \text{ or } B) = P(A) + P(B)$ when events A and B are mutually exclusive.

General Addition Rule: $P(A \text{ or } B) = P(A) + P(B) - P(A \ \& \ B)$ when A and B are not necessarily mutually exclusive.

Complement Rule: $P(E) = 1 - P(\text{not } E)$

example: Roll a die...

A = event that a 3 is rolled

B = event that a 2 is rolled

C = event that a number less than 3 is rolled

$P(A) =$ _____

$P(A \text{ or } B) =$ _____

$P(B) =$ _____

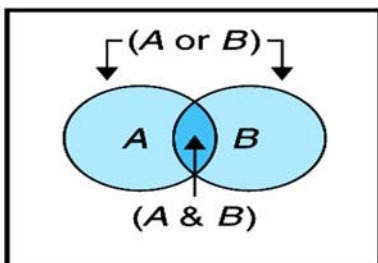
$P(\text{not } A) =$ _____

$P(C) =$ _____

$P(B \text{ or } C) =$ _____

Example: A card is chosen at random from a deck of 52 cards.

Find the probability of choosing a heart or a queen.



4.3 Multiplication Rules and Conditional Probability

Contingency tables give a frequency distribution for cross-classified data. The boxes inside are each called **cells**.

example: The following contingency table provides a cross-classification of U.S. hospitals by type and number of beds:

		24- beds	25-74 beds	75+ beds	
TYPE		B1	B2	B3	
General	H1	260	1586	3557	5403
Psychiatric	H2	24	242	471	737
Chronic	H3	1	3	22	26
Tuberculosis	H4	0	2	2	4
Other	H5	25	177	208	410
		310	2010	4260	6580

a) Describe each of the following **in words**:

H2

B2

(H2 & B2)

(H2 or B2)

b) Compute the probability of each above.

P(H2)=_____

P(B2)=_____

P(H2&B2)=_____

P(H2 or B2)=_____

d) Construct a joint probability distribution:

		24-	25-74	75+	
TYPE		B1	B2	B3	
General	H1				
Psychiatric	H2				
Chronic	H3				
Tuberculosis	H4				
Other	H5				
					1.000

The conditional probability of an event A, given that B occurs, is given by $P(A|B) = \frac{P(A \& B)}{P(B)}$.

example: Roll a die... **A**= the event that a 3 is rolled
B= the event that an odd is rolled

P(A)=_____ P(B)=_____ P(A&B)=_____

P(A or B)=_____ P(A|B)=_____ P(B|A)=_____

example: The table below provides a joint probability distribution for the members of the 105th Congress by legislative group and political party.

		House	Senate	
		C1	C2	
Democrats	P1	0.385	0.084	0.469
Republicans	P2	0.424	0.103	0.527
Other	P3	0.004	0.000	0.004
		0.813	0.187	1.000

If a member of the 105th Congress is selected at random, what is the probability that the member obtained

- a) is a senator?
- b) is a Republican senator?
- c) is a Republican, given that he or she is a senator?
- d) is a senator, given that he or she is a Republican?

Class Example:

	Male	Female	Total
Chocolate			
Strawberry			
Vanilla			
Total			

Multiplication Rule: $P(A \& B) = P(A) * P(B|A)$

example: In Mr. Toner's math class, the male/female ratio is 17:23. Select 2 students at random.

Assume that the first student chosen is not allowed to be chosen a second time. Find the probability of selecting a girl first, then a guy second.

Draw and label a tree diagram for the experiment.

Example: A bag contains 3 red and 4 white marbles. Choose 2 marbles out, one at a time. Draw a tree diagram for this problem both with replacement and without replacement.

with replacement:

without replacement:

What is the difference between independent and dependent trials?

When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability is changed, the events are said to be **dependent events**.

Here are some examples of dependent events:

Drawing a card from a deck, not replacing it, and then drawing a second card.

Selecting a ball from an urn, not replacing it, and then selecting a second ball.

Being a lifeguard and getting a suntan.

Having high grades and getting a scholarship.

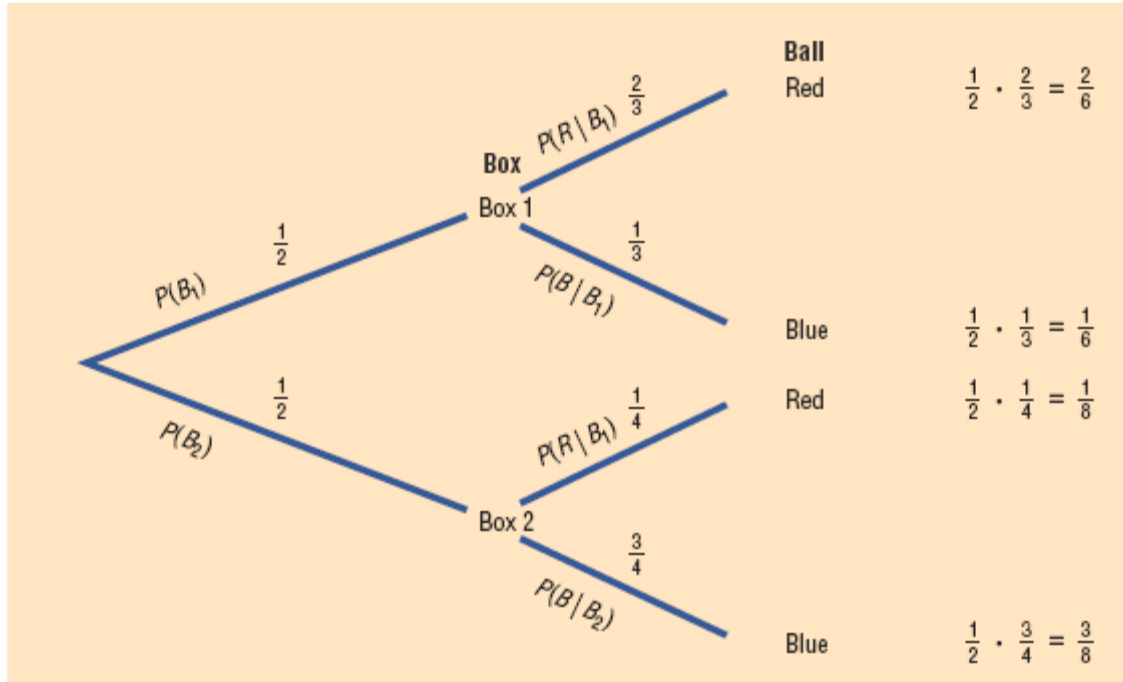
Parking in a no-parking zone and getting a parking ticket.

Example 4-31 on page 216:

Box 1 contains 2 red balls and 1 blue ball. Box 2 contains 3 blue balls and 1 red ball. A coin is tossed. If it falls heads up, box 1 is selected and a ball is drawn. If it falls tails up, box 2 is selected and a ball is drawn. Find the probability of selecting a red ball.

Solution

With the use of a tree diagram, the sample space can be determined as shown in Figure 4–6. First, assign probabilities to each branch. Next, using the multiplication rule, multiply the probabilities for each branch.



Finally, use the addition rule, since a red ball can be obtained from box 1 or box 2.

$$P(\text{red}) = \frac{2}{6} + \frac{1}{8} = \frac{8}{24} + \frac{3}{24} = \frac{11}{24}$$

(Note: The sum of all final probabilities will always be equal to 1.)

Example: A coin is flipped six times. Find the probability that at least one of the flips will contain a tails.

4.4 Counting Rules

Fundamental Counting Rule- When 2 events are to take place in a definite order, with m_1 possibilities for the first event and m_2 possibilities for the second event, then there are $m_1 \cdot m_2$ possibilities altogether. In general, for k events, multiply $m_1 \cdot m_2 \cdots m_k$

example: license plate

Factorial notation:

You can find the factorial, permutation, and combination keys on your TI-83 in the **MATH PROB** menu.

Permutation- a collection or arrangement of objects in which _____ is important.

The number of permutations of r objects from a group of n objects is given by the formula

$${}_n P_r = \frac{n!}{(n-r)!}$$

examples:

1.

2.

3.

Example 4–49**Committee Selection**

In a club there are 7 women and 5 men. A committee of 3 women and 2 men is to be chosen. How many different possibilities are there?

Solution

Here, you must select 3 women from 7 women, which can be done in ${}_7C_3$, or 35, ways. Next, 2 men must be selected from 5 men, which can be done in ${}_5C_2$, or 10, ways. Finally, by the fundamental counting rule, the total number of different ways is $35 \cdot 10 = 350$, since you are choosing both men and women. Using the formula gives

$${}_7C_3 \cdot {}_5C_2 = \frac{7!}{(7-3)!3!} \cdot \frac{5!}{(5-2)!2!} = 350$$

Selected exercises from page 233:

8. **Radio Station Call Letters** The call letters of a radio station must have 4 letters. The first letter must be a K or a W. How many different station call letters can be made if repetitions are not allowed? If repetitions are allowed?

18. **Inspecting Restaurants** How many different ways can a city health department inspector visit 5 restaurants in a city with 10 restaurants?

19. How many different 4-letter permutations can be formed from the letters in the word *decagon*?

- 20. Cell Phone Models** A particular cell phone company offers 4 models of phones, each in 6 different colors and each available with any one of 5 calling plans. How many combinations are possible?
- 32. Selecting Christmas Presents** If a person can select 3 presents from 10 presents under a Christmas tree, how many different combinations are there?
- 36. Freight Train Cars** In a train yard there are 4 tank cars, 12 boxcars, and 7 flatcars. How many ways can a train be made up consisting of 2 tank cars, 5 boxcars, and 3 flatcars? (In this case, order is not important.)
- 44. Charity Event Participants** There are 16 seniors and 15 juniors in a particular social organization. In how many ways can 4 seniors and 2 juniors be chosen to participate in a charity event?

4.5 Probability and Counting Rules

Example 4–52

Magazines

A store has 6 *TV Graphic* magazines and 8 *Newstime* magazines on the counter. If two customers purchased a magazine, find the probability that one of each magazine was purchased.

Solution

$$P(1 \text{ TV Graphic and } 1 \text{ Newstime}) = \frac{{}_6C_1 \cdot {}_8C_1}{{}_{14}C_2} = \frac{6 \cdot 8}{91} = \frac{48}{91}$$

Selected exercise from page 241:

- 12. Selecting Books** Find the probability of selecting 3 science books and 4 math books from 8 science books and 9 math books. The books are selected at random.

In problem #44 (above), what is the probability that 4 seniors and 2 juniors are the six chosen to participate in a charity event?

5.1 Probability Distributions

A discrete random variable is a random variable whose possible values form a discrete data set, only taking on certain values.

example:

# of rooms in a home	x	1	2	3	4	5
	P(x)	0.054	0.173	0.473	0.281	0.020

Find $P(x=3)=$ _____.

A **discrete probability distribution** consists of the values a random variable can assume and the corresponding probabilities of the values. The probabilities are determined theoretically or by observation.

In the next example you are given frequencies, rather than probabilities:

example: The following table displays a frequency distribution for the enrollment by grade in public secondary schools. Frequencies are in thousands of students.

Grade	9	10	11	12
Frequency	3604	3131	2749	2488

Suppose a student in secondary school is to be selected at random. Let x denote the grade level of the student chosen. Determine $P(x=10)$ and interpret your results in terms of percentages.

Two Requirements for a Probability Distribution

1. The sum of the probabilities of all the events in the sample space must equal 1; that is, $\Sigma P(X) = 1$.
2. The probability of each event in the sample space must be between or equal to 0 and 1. That is, $0 \leq P(X) \leq 1$.

5.2 Mean, Variance and Expectation

The mean of a probability distribution is given the special name expected value, defined by $\mu_x = (\Sigma (x \cdot p(x)))$. This means that for a large number of observations of the random variable x , the mean (or expected value) will be approximately μ_x .

example: The following is a probability distribution for the number of customers waiting at Benny's Barber Shop in Cleveland:

TI-83 Procedure:

x	p(x)
0	0.424
1	0.161
2	0.134
3	0.111
4	0.093
5	0.077

Interpretation: If we were to enter the barber shop a large number of times, we would expect approximately 1.519 people to be waiting in line. Could this happen? Explain.

What is the meaning of the standard deviation in this context? It measures the dispersion of the possible values of x relative to the mean. In the example above, we'd expect 1.519 people waiting in line at the barber shop with a standard deviation of 1.674 people.

example: Suppose a lottery contest allowed you to spin a wheel for a prize. On the wheel, each outcome is equally-likely. Find the expected winnings and standard deviation if the prizes are distributed as follows...

Answers and Interpretation:

Prize x	p(x)
\$250	0.01
\$175	0.04
\$150	0.08
\$100	0.12
\$75	0.25
\$50	0.50

Pg. 264-265, example 5-12:

One thousand tickets are sold at \$1 each for a color television valued at \$350. What is the expected value of the gain if a person purchases one ticket?

Solution

The problem can be set up as follows:

	Win	Lose
Gain X	\$349	-\$1
Probability $P(X)$	$\frac{1}{1000}$	$\frac{999}{1000}$

Two things should be noted. First, for a win, the net gain is \$349, since you do not get the cost of the ticket (\$1) back. Second, for a loss, the gain is represented by a negative number, in this case -\$1. The solution, then, is

$$E(X) = \$349 \cdot \frac{1}{1000} + (-\$1) \cdot \frac{999}{1000} = -\$0.65$$

Expected value problems of this type can also be solved by finding the overall gain (i.e., the value of the prize won or the amount of money won, not considering the cost of the ticket for the prize or the cost to play the game) and subtracting the cost of the tickets or the cost to play the game, as shown:

$$E(X) = \$350 \cdot \frac{1}{1000} - \$1 = -\$0.65$$

Here, the overall gain (\$350) must be used.

Note that the expectation is -\$0.65. This does not mean that you lose \$0.65, since you can only win a television set valued at \$350 or lose \$1 on the ticket. What this expectation means is that the average of the losses is \$0.65 for each of the 1000 ticket holders. Here is another way of looking at this situation: If you purchased one ticket each week over a long time, the average loss would be \$0.65 per ticket, since theoretically, on average, you would win the set once for each 1000 tickets purchased.

Example:

5.3 The Binomial Distribution

A **binomial experiment** is a probability experiment that satisfies the following four requirements:

1. There must be a fixed number of trials.
2. Each trial can have only two outcomes or outcomes that can be reduced to two outcomes. These outcomes can be considered as either success or failure.
3. The outcomes of each trial must be independent of one another.
4. The probability of a success must remain the same for each trial.

Examples of binomial trials:

A population in which each member is classified as either having or not having a specific attribute is called a _____ population.

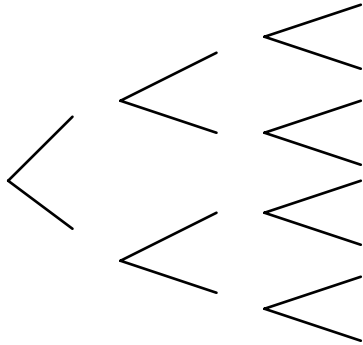
Suppose a survey were done of all U.S. households to see if they own a microwave. The population to be surveyed would be huge! We cannot get exact percentages, but only an estimation.

When running this survey, the sampling could *either* be done **with** or **without** replacement. Suppose you had a huge list containing every person's name in the U.S. If you were to cross off names as you surveyed people, so that you would not call them twice, then you would be surveying **without** replacement.

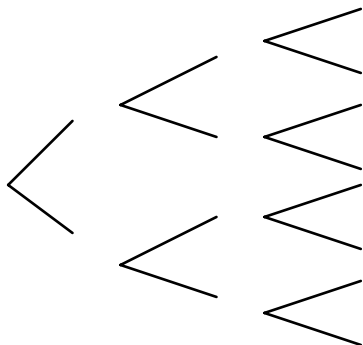
Would it make a difference if you crossed out names if you had a huge list of names and you were doing random sample surveying? Explain.

Rule of thumb: If a sample size is less than 5% of a population size, then Bernoulli (independent) trials may be assumed (and surveying can be done with replacement).

example: Draw a tree diagram for flipping a coin three times.



example: Draw and label a tree diagram for flipping a coin three times if the coin is bent and has a 75% chance of landing on "heads" each time it is flipped. Find and label the sample space and each of the associated probabilities.



Suppose n binomial trials are to be performed. The probability distribution for x successes in n binomial trials is given by $P(x) = \binom{n}{x} p^x (1-p)^{n-x}$,

where $n = \#$ of trials, $x = \#$ of successes, $p =$ probability of a success

On the TI-83, we use the binompdf and binomcdf functions, found in the **DIST** menu:

Binompdf(numtrials, probsuccess, numsuccesses) finds $P(x = \#)$

Binomcdf(numtrials, probsuccess, numsuccesses) finds $P(x \leq \#)$

example: A salesperson makes 8 contacts per day with potential customers. From past experience, we know that the probability a potential customer will purchase a product is 0.10.

a) What is the probability that he/she makes exactly 2 sales on a particular day?

b) What is the probability he/she makes at most 2 sales on a particular day?

c) What is the probability he/she makes at least 2 sales on a particular day?

PATTERNS:

examples:

1. A true/false test has 15 questions on it. If you randomly guess at each question, what is...

a) $P(x=6 \text{ correct})$ _____

b) $P(x>11 \text{ correct})$ _____

c) $P(x<9 \text{ correct})$ _____

2. A 10 question multiple choice test has 5 possible responses for each question. If you randomly guess at each question, what is

a) $P(x=8 \text{ correct})$ _____

b) $P(x \geq 6 \text{ correct})$ _____

b) $P(x \leq 7 \text{ correct})$ _____

example: According to the US Census Bureau, 25% of US children are not living with both parents. If 10 US children are selected at random, determine the probability that the number not living with both parents is...

a) exactly two. _____

b) at most two. _____

c) between three and six, inclusive. _____

Binomial Expected Values

example: As reported by Television Bureau of Advertising, Inc., in *Trends in Television*, 84.2% of U.S. households have a VCR. If six households are randomly selected without replacement, what is the (approximate) probability that the number of households sampled that have a VCR will be

1. exactly four?
2. at least four?
3. At most five?
4. Between two and five, inclusive?
5. Determine the (approximate) probability distribution of the random variable Y , the number of households of the six sampled that have a VCR.
6. Determine and interpret the mean of the random variable Y .
7. Obtain the standard deviation of Y .

5.4 The Poisson Distribution

A type of probability distribution that is often useful in describing the number of events that will occur in a specific amount of time or in a specific area or volume is the **Poisson distribution**. Typical examples of random variables for which the Poisson probability distribution provides a good model are:

1. The number of traffic accidents per month in a busy intersection.
2. The number of noticeable surface defects (scratches, dents, etc.) found by quality inspectors on a new automobile.
3. The parts per million of some toxin found in the water or air emission from a manufacturing plant.
4. The number of diseased trees per acre of a certain woodland.
5. The number of death claims received per day by an insurance company.
6. The number of unscheduled admissions per day to a hospital.

Characteristics of a Poisson Random Variable

1. The experiment consists of counting the number of times a certain event occurs during a given unit of time or in a given area or volume (or weight, distance, or any other unit of measure).
2. The probability that an event occurs in a given unit of time, area, or volume is the same for all the units.
3. The number of events that occur in one unit of time, area, or volume is independent of the number that occur in other units.
4. The mean (or expected) number of events in each unit is denoted by the Greek letter, lambda, λ , and the standard deviation is $\sqrt{\lambda}$.

The characteristics of the Poisson random variable are usually difficult to verify for practical examples. The examples given satisfy them well enough that the Poisson distribution provides a good model in many instances. As with all probability models, the real test of the adequacy of the Poisson model is in whether it provides a reasonable approximation to reality- that is, whether empirical data support it.

The Poisson Distribution is used to model the frequency with which an event occurs during a particular period of **time** using $p(x) = e^{-\lambda} \left(\frac{\lambda^x}{x!} \right)$, where λ (lambda) is given and $e \approx 2.71828$. The expected value of a Poisson distribution is given by $\mu_x = \lambda$, with $\sigma_x = \sqrt{\lambda}$.

On the TI-83 **DIST** menu you can find poissonpdf and poissoncdf.

example: The owner of a fast food restaurant knows that, on the average, 2.4 cars (customers) use the drive-through window between 3:00 pm and 3:15 pm. Assuming that the number of such cars has a Poisson distribution, find the probability that, between 3:00 pm and 3:15 pm,

a) exactly two cars will use the drive-through window.

b) at least three cars will use the drive-through window.

Table 5–1 Summary of Discrete Distributions

1. Binomial distribution

$$P(X) = \frac{n!}{(n - X)!X!} \cdot p^X \cdot q^{n-X}$$

$$\mu = n \cdot p \quad \sigma = \sqrt{n \cdot p \cdot q}$$

Used when there are only two independent outcomes for a fixed number of independent trials and the probability for each success remains the same for each trial.

2. Multinomial distribution

$$P(X) = \frac{n!}{X_1! \cdot X_2! \cdot X_3! \cdot \dots \cdot X_k!} \cdot p_1^{X_1} \cdot p_2^{X_2} \cdot \dots \cdot p_k^{X_k}$$

where

$$X_1 + X_2 + X_3 + \dots + X_k = n \quad \text{and} \quad p_1 + p_2 + p_3 + \dots + p_k = 1$$

Used when the distribution has more than two outcomes, the probabilities for each trial remain constant, outcomes are independent, and there are a fixed number of trials.

3. Poisson distribution

$$P(X; \lambda) = \frac{e^{-\lambda} \lambda^X}{X!} \quad \text{where} \quad X = 0, 1, 2, \dots$$

Used when n is large and p is small, the independent variable occurs over a period of time, or a density of items is distributed over a given area or volume.

4. Hypergeometric distribution

$$P(X) = \frac{{}_a C_X \cdot {}_b C_{n-X}}{{}_{a+b} C_n}$$

Used when there are two outcomes and sampling is done without replacement.

Probability Review Problems

1. On a quiz consisting of 3 true/false questions, an unprepared student must guess at each one. The guesses will be random.
 - A. List the different possible solutions.

 - B. What is the probability of answering all 3 questions correctly?

 - C. What is the probability of guessing incorrectly for all questions?

 - D. What is the probability of passing the quiz by guessing correctly for at least 2 questions?

2.
 - A. If a person is randomly selected, find the probability that his or her birthday is October 18, which is National Statistics Day in Japan. Ignore leap years.

 - B. If a person is randomly selected, find the probability that his or her birthday is in November. Ignore leap years.

3. After collecting IQ scores from hundreds of subjects, a boxplot is constructed with this 5-number summary: {82, 91, 100, 109, 118}. If one of the subjects is randomly selected, find the probability that his or her IQ score is greater than 109.

4. Find the probability of getting 4 consecutive aces when 4 cards are drawn without replacement from a shuffled deck.

5. A typical “combination” lock is opened with the correct sequence of 3 numbers between 0 and 49 inclusive. How many different sequences are possible? (A number can be used more than once.) Are these sequences combinations or are they actually permutations?

6. Mars, Inc., claims that 20% of its plain M&M candies are red. Find the probability that when 15 plain M&M candies are randomly selected, exactly 20% (or 3 candies) are red.

7. Suppose that a city has two hospitals. Hospital A has about 100 births per day, while Hospital B has only about 20 births per day. Assume that each birth is equally likely to be a boy or a girl. Suppose that for one year you count the number of days on which the a hospital has 60% or more of that day’s births turn out to be boys. Which hospital would you expect to have more such days? Explain your reasoning.

8. a. If $P(A \text{ or } B) = \frac{1}{3}$, $P(B) = \frac{1}{4}$, and $P(A \text{ and } B) = \frac{1}{5}$, find $P(A)$.

- b. If $P(A) = 0.4$ and $P(B) = 0.5$, what is known about $P(A \text{ or } B)$ if A and B are mutually exclusive events?