The Fundamental Theorem of Calculus
If a function f is continuous on $[a,b]$ and F is an antiderivative of f on the interval $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

This theorem states that ***provided you can find*** an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.

Notation: $\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)$

For example, to evaluate $\int_1^3 x^3 dx$, we would write:

$$\int_1^3 x^3 dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20$$

It is not necessary to use a constant of integration C because it "cancels out":

$$\int_a^b f(x)dx = \left[F(x) + C\right]_a^b$$
$$= \left[F(b) + C\right] - \left[F(a) + C\right]$$
$$= F(b) - F(a)$$

Mean Value Theorem for Integrals
If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that $\int_a^b f(x)dx = f(c)(b-a)$.

This is an existence theorem. It does not help you find c; it merely guarantees the existence of at least one number c in the interval.

The value $f(c)$, given by the Mean Value Theorem for Integrals is called the **average value of f on** $[a,b]$.

If f is integrable on $[a,b]$, then the average value of f on the interval is $\frac{1}{b-a}\int_a^b f(x)dx$.

The Second Fundamental Theorem of Calculus
If f is continuous on an open interval I containing c, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x).$$

Notice how the upper limit of integration is x. A different variable is used for integration as a "substitute" variable.

Problems - pages 291-294
In exercises 5-26, evaluate the definite integral.

6. $\int_2^7 3dv$

10. $\int_1^3 \left(3x^2 + 5x - 4\right)dx$

16. $\int_{-3}^3 v^{1/3} dv$

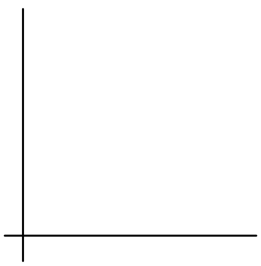
22. $\int_{-8}^{-1} \frac{x - x^2}{2\sqrt[3]{x}}dx$

26. $\int_0^4 \left|x^2 - 4x + 3\right| dx$

30. $\int_{\pi/4}^{\pi/2} \left(2 - \csc^2 x\right) dx$

36. Determine the area of the indicated region.

$y = \dfrac{1}{x^2}$



42. Find the area of the region bounded by the graphs of the equations.

$$\begin{cases} y = -x^2 + 3x \\ y = 0 \end{cases}$$

In exercises 43-46, find the value of ***c*** guaranteed by the Mean Value Theorem for Integrals for the function over the interval.

44. $f(x) = \dfrac{9}{x^3}$ **on** $[1,3]$

46. $f(x) = \cos x$ **on** $\left[\dfrac{-\pi}{3}, \dfrac{\pi}{3}\right]$

48. Find the average value of the function over the interval and all values of ***x*** in the interval for which the function equals its average value.

$$f(x) = \frac{4\left(x^2 + 1\right)}{x^2} \quad \text{on } [1,\ 3]$$

50. Find the average value of the function over the interval and all values of x in the interval for which the function equals its average value.

$$f(x) = \cos x \quad \text{on } \left[0, \frac{\pi}{2}\right]$$

62. The velocity v of the flow of blood at a distance r from the central axis of an artery of radius R is $v = k\left(R^2 - r^2\right)$, where k is the constant of proportionality. Find the average rate of flow of blood along a radius of the artery. (Use 0 and R as the limits of integration.)

Ex. Use the Second Fundamental Theorem of Calculus to find $F'(x)$.

$$F(x) = \int_0^x \tan^4 t \, dt$$

90. Find $F'(x)$, given $F(t) = \int_2^{x^2} \frac{1}{t^3} dt$

Consider a particle moving along the x-axis where $x(t)$ is the position of the particle at time t, $x'(t)$ is its velocity, and $\int_a^b |x'(t)| dt$ is the distance the particle travels in the interval of time.

97. The position function is $x(t) = t^3 - 6t^2 + 9t - 2$ where $0 \le t \le 5$. Find the total distance the particle travels in 5 units of time.

Section 4.5 INTEGRATION BY SUBSTITUTION

Antidifferentiation of a Composite Function

$\int f(g(x))g'(x)dx = F(g(x)) + C$

If $u=g(x)$, then $du = g'(x)$ and $\int f(u)du = F(u) + C$.

General Power Rule

$\int [g(x)]^n g'(x)dx = \frac{[g(x)]^{n+1}}{n+1} + C$, $n \neq -1$

$\int u^n du = \frac{u^{n+1}}{n+1} + C$, $n \neq -1$

[Go over page 292, example 7 in text.]

Change of Variables

$\int_a^b f(g(x))dx = \int_{g(a)}^{g(b)} f(u)du$

Integration of Even/Odd Functions

If f is an even function, $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

If f is an odd function, $\int_{-a}^{a} f(x)dx = 0$.

Problems - pages 304-306

For problems 1-6, find u and du for the integral.

2. $\int x^2\sqrt{x^3+1}\,dx$

4. $\int \sec 2x \tan 2x dx$

6. $\int \frac{\cos x}{\sin^2 x}dx$

For problems 7-34, evaluate the indefinite integral and check the result by differentiation.

8. $\int (x^2-9)^3(2x)dx$

10. $\int \sqrt[3]{1-2x^2}(-4x)dx$

14. $\int x(4x^2+3)^3 dx$

18. $\int u^2\sqrt{u^3+2}du$

22. $\int \frac{x^2}{(16-x^3)^2}dx$

28. $\int \frac{1}{2\sqrt{x}}dx$

30. $\int \frac{t+2t^2}{\sqrt{t}}dt$

34. $\int 2\pi y\left(8 - y^{3/2}\right)dy$

In exercises 35-38, solve the differential equation.

36. $\dfrac{dy}{dx} = \dfrac{10x^2}{\sqrt{1+x^3}}$

38. $\dfrac{dy}{dx} = \dfrac{x-4}{\sqrt{x^2-8x+1}}$

In exercises 43-56, evaluate the indefinite integral.

48. $\int x \sin x^2 dx$

50. $\int \sec(1-x)\tan(1-x)dx$

52. $\int \sqrt{\tan x}\sec^2 x dx$

58. Find an equation for the function f that has the derivative $f'(x) = \pi \sec \pi x \tan \pi x$ and whose graph passes through $\left(\frac{1}{3}, 1\right)$.

In exercises 63-70, evaluate the indefinite integral by the method shown in example 5.

64. $\int x\sqrt{2x+1}\, dx$

66. $\int (x+1)\sqrt{2-x}\, dx$

68. $\int \frac{2x-1}{\sqrt{x+3}}\, dx$

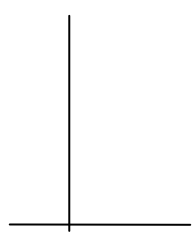
For exercises 71-82, evaluate the definite integral.

74. $\int_{0}^{1} x\sqrt{1-x^2}\, dx$

78. $\int_{0}^{2} x\sqrt[3]{4+x^2}\, dx$

82. $\int_{\pi/3}^{\pi/2} \left(x+\cos x\right) dx$

88. Find the area of the region. $\int_{-2}^{6} x^2\sqrt[3]{x+2}\, dx$



100. Integrate $\int \sin x \cos x dx$ in two different ways. Explain any differences in the form of the answers.

106. Use the symmetry of the graphs of the sine and cosine functions as an aid in evaluating each of the integrals.

a. $\int_{-\pi/4}^{\pi/4} \sin x dx$

b. $\int_{-\pi/4}^{\pi/4} \cos x dx$

c. $\int_{-\pi/2}^{\pi/2} \cos x dx$

d. $\int_{-\pi/2}^{\pi/2} \sin x \cos x dx$

ex. The minimum stockpile level of gasoline in the United States can be approximated by the model $Q = 217 + 13\cos\frac{\pi(t-3)}{6}$ where Q is measured in millions of barrels of gasoline and t is the time in months, with $t=1$ corresponding to January. Fine the average minimum level given by this model during the following periods.

a. the first quarter $(0 \le t \le 3)$.

b. the second quarter $(3 \le t \le 6)$

c. The entire year $(0 \le t \le 12)$

Some elementary functions simply do not have antiderivatives that are elementary functions. If you need to evaluate a definite integral involving a function whose antiderivative cannot be found, the Fundamental Theorem of Calculus cannot be applied, and you must resort to an approximation technique.

One way to approximate a definite integral is to use ***n*** trapezoids, as shown in this picture:

Area of **i**th trapezoid:

$$\left[\frac{f(x_{i-1}) + f(x_i)}{2}\right]\left(\frac{b-a}{n}\right)$$

The Trapezoid Rule

Let ***f*** be continuous on $[a, b]$. The Trapezoidal Rule for approximating $\int_a^b f(x)dx$ is given by

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

As ***n*** approaches infinity, the approximation approaches $\int_a^b f(x)dx$.

Simpson's Rule (***n*** must be even)

Let ***f*** be continuous on $[a, b]$. The Simpson's Rule for approximating $\int_a^b f(x)dx$ is given by

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 4f(x_{n-1}) + f(x_n)\right]$$ coeff: 142424…4241

As ***n*** approaches infinity, the approximation approaches $\int_a^b f(x)dx$.

Errors in the Trapezoidal Rule and Simpson's Rule

- If ***f*** has a continuous second derivative on $[a, b]$, then the error ***E*** is in approximating $\int_a^b f(x)dx$ by the Trapezoidal Rule is $E \le \frac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$, $\quad a \le x \le b$.

- If ***f*** has a continuous fourth derivative on $[a, b]$, then the error ***E*** is in approximating $\int_a^b f(x)dx$ by Simpson's Rule is $E \le \frac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right]$, $\quad a \le x \le b$.

In exercises 1-10, use the Trapezoidal Rule and Simpson's Rule to approximate the value of the definite integral for the indicated value of ***n***. Round your answer to four decimal places and compare the resuls with the exact value of the definite integral.

2. $\int_0^1 \left(\frac{x^2}{2} + 1\right) dx, \quad n = 4$

6. $\int_0^8 \sqrt[3]{x}\, dx, \quad n = 8$

25. Use the error formulas to find the maximum possible error in approximating $\int_0^1 \frac{1}{x+1}\, dx$, with ***n*=4** using (a) the Trapezoidal Rule and (b) Simpson's Rule.

$$E \le \frac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right], \qquad a \le x \le b$$

$$E \le \frac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right], \qquad a \le x \le b$$

30. Use the error formulas to find n such that the error in approximating $\int_0^1 \frac{1}{x+1}\,dx$ is less than **0.00001** by using (a) the Trapezoidal Rule and (b) Simpson's Rule.

$$E \le \frac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right], \quad a \le x \le b$$

$$E \le \frac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right], \quad a \le x \le b$$