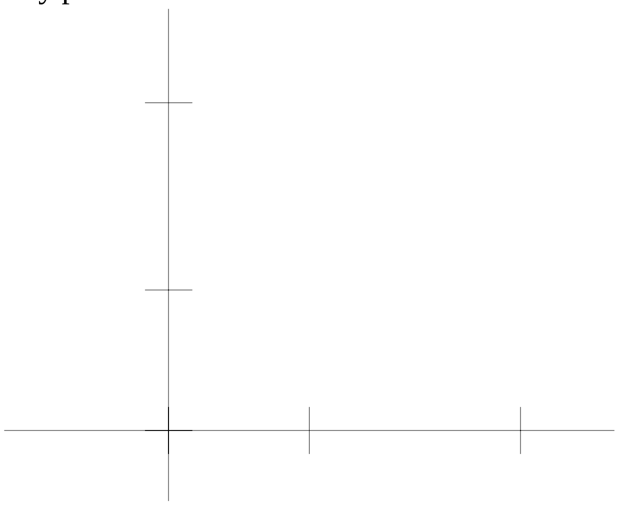
Math 226 Notes 1.1 A PREVIEW OF CALCULUS

There are 2 main problems in calculus:

1. Finding a tangent line to a curve though a point on the curve.
2. Finding the area under a curve on some interval.

SEE and DISCUSS the pictures on pages 42-43 in your text.

Key picture:



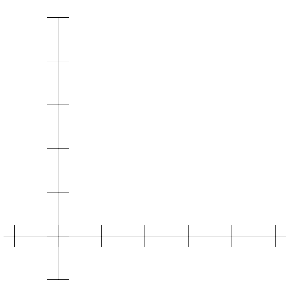
As ***Q*** approaches ***P***, the secant line approaches the tangent line. As Δx gets smaller, ***Q*** gets closer to ***P*** and the slope of the tangent line can be found. We would then use the point-slope formula to actually obtain the equation of the tangent line.

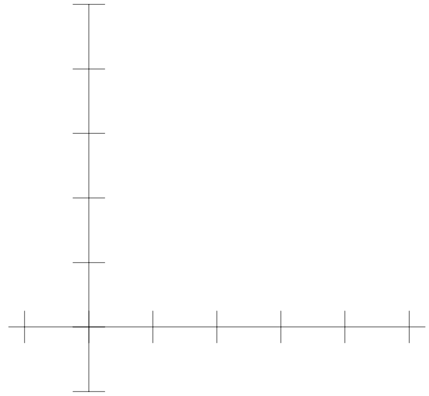
$$m_{\sec} = \frac{f(c+\Delta x) - f(c)}{(c+\Delta x) - c}$$
$$= \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

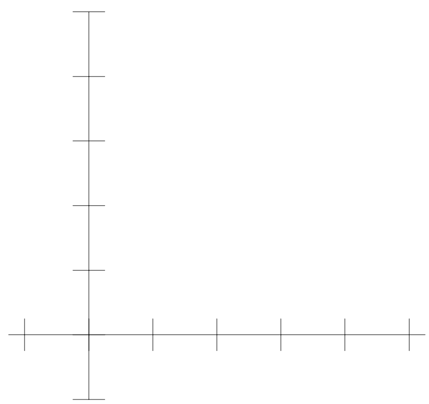
To find the area under a curve, we use "representative rectangles."

Example: page 46 Exploration

Find the area under $y = x^2$ on the interval $(0,1)$.







Math 226 Notes Section 1.2

INTRODUCTION TO LIMITS

The "**limit as x approaches c**" (from either side) is written: $\lim_{x \to c} f(x) = L$. A **limit** is defined as the value approached from the left or right, regardless of the value of the function at that point.

There are 3 ways in which "No Limit Exists":

1. $f(x)$ approaches different values from each side.
2. $f(x)$ approaches $\pm\infty$ on either or both sides.
3. $f(x)$ oscillates between two fixed values.

Pictures:

Methods of Finding Limits:

1. Look at the graph.
2. Use an x-y chart if the graph is hard to create or read.
3. Use the formal method (ε and δ).
4. Use other numerical techniques.

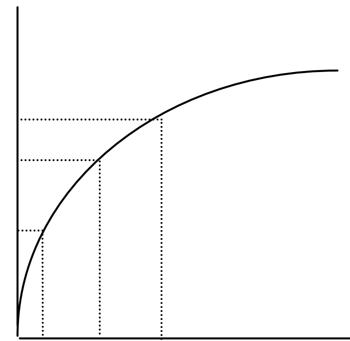
Page 55 – Examine and discuss graphs for exercises 13, 14, 17, 20 and 24.

Notes:

Formal Definition of a Limit

Let f be a function defined on an open interval containing **c** (except possibly at **c**) and let L be a real number. $\lim_{x \to c} f(x) = L$ means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Picture:



problems - page 55

36. $\lim_{x \to 5} (x^2 + 4)$

a) Find the limit.

b) Find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

34. $\lim_{x \to 4}\left(4 - \frac{x}{2}\right)$

a) Find the limit.

b) Find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

In problems 37-48, find the limit L. Then use the $\varepsilon - \delta$ definition to prove that the limit is L.

42. $\lim_{x \to 2}(-1)$

38. $\lim_{x \to -3}(2x+5)$

44. $\lim_{x \to 4}\left(\sqrt{x}\right) = 2$

Notes Section 1.3 FINDING LIMITS ANALYTICALLY

Some limits can be found by direct substitution. This happens when the limit as **x** approaches **c** actually equals $f(c)$.

For constant functions, $\lim_{x \to c} b = b$.

For the identity function, $\lim_{x \to c} x = c$.

For power functions, $\lim_{x \to c} x^n = c^n$.

In fact, for <u>all</u> polynomial, trigonometric, and rational functions $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq o$, you can plug in c in order to find the limit.

<u>Properties:</u>
If **b** and **c** are real numbers and **n** is a positive integer, and if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$, then…

1. $\lim_{x \to c} [bf(x)] = bL$ scalar multiplication
2. $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$ sum or difference
3. $\lim_{x \to c} [f(x) \cdot g(x)] = LK$ product
4. $\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{K}$ quotient $g(x) \neq 0$
5. $\lim_{x \to c} [f(x)]^n = L^n$ power

<u>For composite functions,</u>

1. If $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then $\lim_{x \to c} f(g(x)) = f(L)$.
2. If two functions, $f(x) = g(x)$ for all $x \neq c$, then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$

If you try to evaluate a limit and get $\frac{0}{0}$, called an **indeterminate form**, you can either try to
1. factor and cancel, or
2. rationalize the <u>numerator</u>.

<u>Two Special Trig Limits:</u>
1. $\lim_{x \to 0} \left(\frac{\sin x}{x}\right) = 1$
2. $\lim_{x \to 0} \left(\frac{1 - \cos x}{x}\right) = 0$

<u>The Squeeze Theorem:</u>
If you have a hard time finding a limit, you can sometimes find the limit by "squeezing" it between two functions with known limits.

If $h(x) \leq f(x) \leq g(x)$ for all **x** in an open interval containing **c**, except possibly at **c** itself, and if $\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$, then $\lim_{x \to c} f(x)$ exists and equals ***L*** too.

Problems - pages 67-69

10. Find $\lim_{x \to 1} \left(-x^2 + 1\right)$

14. Find $\lim_{x \to -3} \left(\frac{2}{x + 2}\right)$

32. Find $\lim_{x \to \pi} (\cos 3x)$

50. Find $\lim_{x \to 2} \left(\frac{2 - x}{x^2 - 4}\right)$

54. Find $\lim_{x \to 0} \left(\frac{\sqrt{2+x} - \sqrt{2}}{x} \right)$

60. Find $\lim_{\Delta x \to 0} \left(\frac{(x + \Delta x)^2 - x^2}{\Delta x} \right)$

68. $\lim_{x \to 0} \left(\frac{3(1 - \cos x)}{x} \right)$

72. $\lim_{x \to 0} \left(\frac{\tan^2 x}{x} \right)$

78. $\lim_{x \to 0} \left(\frac{\sin 2x}{\sin 3x} \right)$

88. Use the Squeeze Theorem to find $\lim_{x \to c} f(x)$.

$c = a$

$b - |x - a| \le f(x) \le b + |x - a|$

Notes Section 1.4 CONTINUITY AND ONE-SIDED LIMITS

If a function is **continuous** at **x=c**, there is no interruption in the graph of ***f*** at **c**.

A function ***f*** is **continuous** at **x=c** if the following three conditions are met:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$.

A function is **continuous on an open interval** (a, b) if it is continuous at each point in that interval. A function that is continuous on $(-\infty, +\infty)$ is said to be **continuous everywhere**.

There are 2 types of discontinuities:

1. removable

2. non-removable

The Existence of a Limit

Let ***f*** be a function and let **c** and **L** be real numbers. The limit of $f(x)$ as **x** approaches **c** is **L** if and only if

$\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$.

This definition opens the door to one-sided limits and allows us to define continuity on a closed interval.

A function ***f*** is continuous on a closed interval $[a, b]$ if it is continuous on the open interval (a, b) and

$\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

We say "the function is continuous from the right at a" and "continuous from the left at b."

picture:

Properties of Continuity:

If **b** is a real number and ***f*** and ***g*** are continuous functions at **x=c**, then the following functions are also continuous:

1.	bf	scalar multiplication
2.	$f \pm g$	sum or difference
3.	fg	product
4.	$\frac{f}{g}, \ g \neq 0$	quotient

If ***g*** is continuous at **c** and ***f*** is continuous at $g(c)$, then $f \circ g(x) = f(g(x))$ is continuous at **c**.

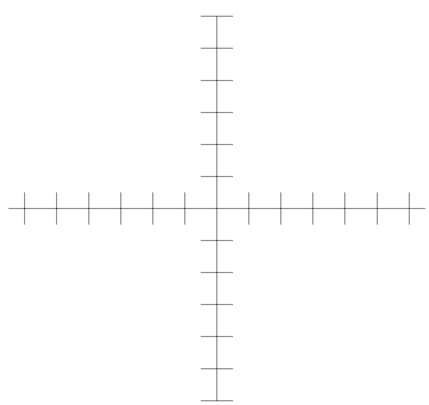
Intermediate Value Theorem:

If ***f*** is continuous on the closed interval $[a, b]$ and **k** is any number between $f(a)$ and $f(b)$, then there exists at least one number **c** in $[a, b]$ such that $f(c) = k$. (This is an existence theorem.)

problems - pages 78-81

8. Find the limit, if it exists: $\lim_{x \to 2^+} \left(\frac{2 - x}{x^2 - 4} \right)$

12. Find the limit, if it exists: $\lim_{x \to 2} \left(\frac{|x - 2|}{x - 2} \right)$



18. Find the limit, if it exists: $\lim_{x \to 1} \begin{cases} x, & x \le 1 \\ 1 - x, & x > 1 \end{cases}$

34. Find the x value(s) at which f is not continuous. Which discontinuities are removable?

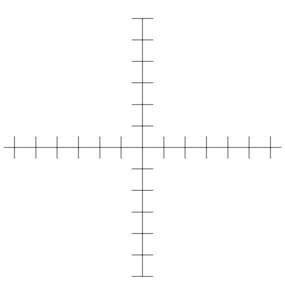
$$f(x) = \frac{1}{x^2 + 1}$$

38. Find the x value(s) at which f is not continuous. Which discontinuities are removable?

$$f(x) = \frac{1}{x^2 - 1}$$

40. Find the x value(s) at which f is not continuous. Which discontinuities are removable?

$$f(x) = \frac{x - 3}{x^2 - 9}$$



58. Find **a** so that $g(x)$ is continuous on the entire real line.

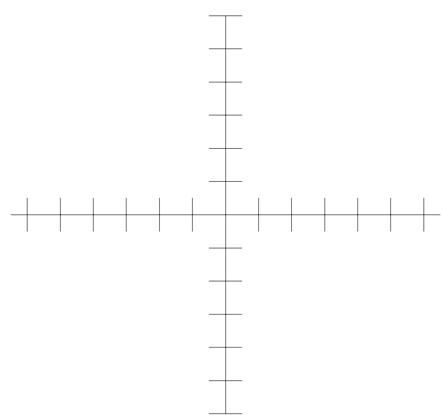
$$g(x) = \begin{cases} \dfrac{4 \sin x}{x}, & x < 0 \\ a - 2x, & x > 0 \end{cases}$$

62. Discuss the continuity of $h(x) = f(g(x))$.

$$f(x) = \frac{1}{\sqrt{x}} \qquad g(x)=x-1$$

70. Find the interval(s) on which the function is continuous.

$$f(x) = x\sqrt{x+3}$$

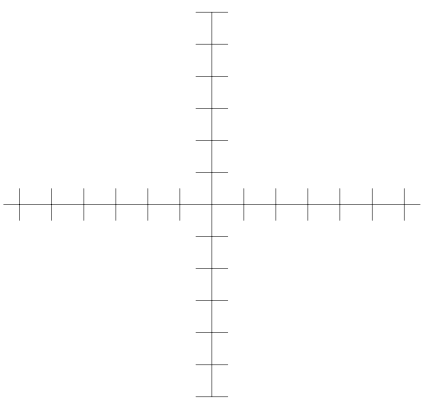


Notes Section 1.5 INFINITE LIMITS

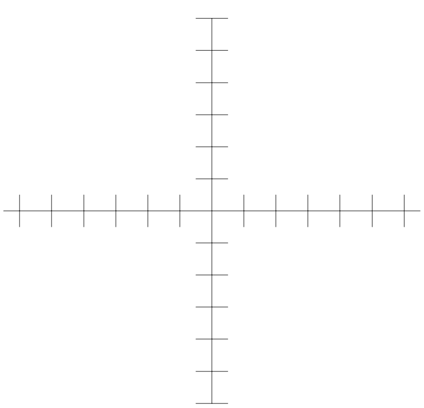
Graphing Rational Functions

1. There is an x-intercept everywhere the numerator equals zero.
2. There is a vertical asymptote whenever the denominator is zero.
3. If the degree of the numerator is *less* than the degree of the denominator, then the x-axis is a horizontal asymptote.
4. If the degree of the numerator *equals* the degree of the denominator, then the **ratio of the leading coefficients** is the horizontal asymptote.
5. If the degree of the numerator is *greater* than the degree of the denominator, then there is no horizontal asymptote.
6. Use an x-y chart to fill in extra points.

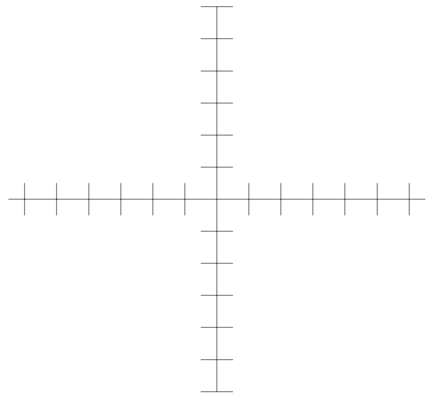
Graph: $y = \frac{x+3}{2x-1}$



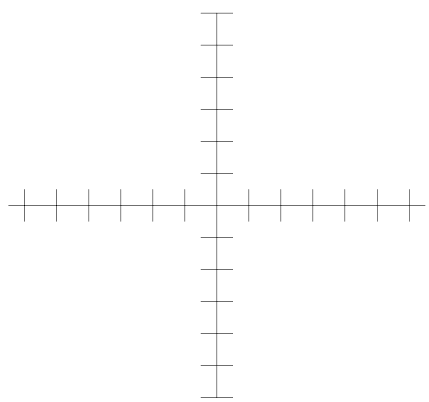
Graph: $y = \frac{1}{(x-2)^2}$



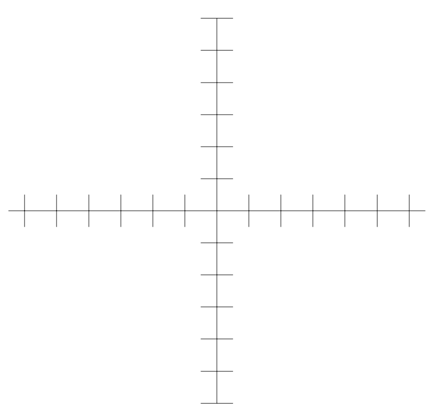
Graph: $y = \frac{x+3}{x^2-9}$



Graph: $y = \frac{x}{x^2-4}$



Graph: $y = \frac{x^2+2x-8}{x^2-4}$



Let f be a function that is defined at every real number in some open interval containing **c** (except possibly at **c** itself).

$\lim_{x \to c} f(x) = \infty$ means that for each $M > 0$, there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$.

$\lim_{x \to c} f(x) = -\infty$ means that for each $N < 0$, there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - c| < \delta$.

<u>Properties:</u>

If $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = L$, then

1. $\lim_{x \to c} [f(x) \pm g(x)] = \infty$
2. $\lim_{x \to c} [f(x) \cdot g(x)] = \begin{cases} \infty & \text{if } L > 0 \\ -\infty & \text{if } L < 0 \end{cases}$
3. $\lim_{x \to c} \left[\dfrac{g(x)}{f(x)}\right] = 0$

problems - page 88

In exercises 33-48, find the vertical asymptotes (if any) of the function,

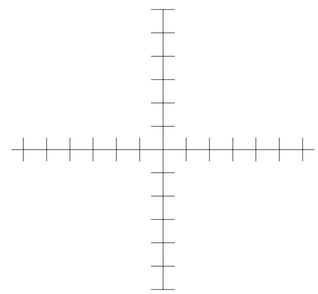
34. $g(x) = \dfrac{2 + x}{1 - x}$

36. $f(x) = \dfrac{x^2}{x^2 + 16}$

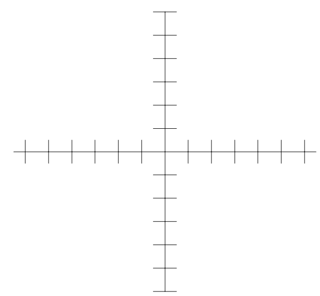
ex. $f(x) = \sec(\pi x)$

Ex. $h(x) = \dfrac{x^2 - 4}{x^3 + 2x^2 + x + 2}$

ex. Find $\lim_{x \to 1^+} \left(\dfrac{2 + x}{1 - x}\right)$



42. Find $\lim_{x \to 0^-} \left(x^2 - \dfrac{1}{x}\right)$



59. A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of ½ revolutions per second. The rate at which the light beam moves along the wall is $r = 50\pi \sec^2 \vartheta$ ft/sec.

a. Find the rate **r** when ϑ is $\dfrac{\pi}{6}$.

b. Find the rate **r** when ϑ is $\frac{\pi}{3}$.

c. Find the rate **r** when as $\vartheta \to \left(\frac{\pi}{2}\right)^-$.

58. *Boyle's Law*. For a quantity of gas at a constant temperature, the pressure P is inversely proportional to the volume V. Find the limit of P as $V \to 0^+$.

Page 90 – True or False? Determine whether the statement is true of false. If it is false, explain why or give an example that shows it is false.

67. If p(x) is a polynomial, then the graph of the function given by $f(x) = \frac{p(x)}{x-1}$ has a vertical asymptote at x=1.

68. The graph of a rational function has a least one vertical asymptote.

69. The graphs of polynomial functions have no vertical asymptotes.

70. If f has a vertical asymptote at x=0, then f is undefined at x=0.

Notes Section 1R CHAPTER ONE REVIEW

problems - page 91

16. Find $\lim_{x \to 0} \left(\frac{\sqrt{4+x} - 2}{x} \right)$

18. Find $\lim_{s \to 0} \left(\frac{\frac{1}{\sqrt{1+s}} - 1}{s} \right)$

Ex. $\lim_{x \to 2} \left(\frac{1}{\sqrt[3]{x^2 - 4}} \right)$

24. $\lim_{\Delta x \to 0} \left(\frac{\cos(\pi + \Delta x) + 1}{\Delta x} \right)$

Ex. $\lim_{x \to 0^+} \left(\frac{\sec x}{x} \right)$

36. (modified) Determine the intervals on which $f(x)$ is continuous. $f(x) = \frac{3x^2 - x - 2}{x - 1}$

42. Determine the intervals on which the function is continuous. $f(x) = \sqrt{\frac{x+1}{x}}$

page 81 - Intermediate Value Theorem

84. Verify that the Intermediate Value Theorem applies to the interval $[0,3]$ with $f(c) = 0$.

$f(x) = x^2 - 6x + 8$